A Higher Dimensional Generalization of Subspace Theorem for Closed Subschemes and Consequences

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Abstract

In this article, we follow [HLX24]'s work. We try to generalize the Diophantine approximation inequality from surface case to a general variety case under some special conditions. As applicationm, we study various Diophantine problems, including the greatest common divisors, degeneracy of integral points. We also state the results in the complex analytic setting.

1 Introduction

In this section, we will give some well-known results in the Diophantine approximation, and we will also show there b-Cartierv divisor's version. Main theorems will be given in [Voj23].

1.1 Subspace theorem

First, let's begin with the basic subspace theorem:

Theorem 1.1 (Schmidt's subspace theorem)**.** *Let k be a number field, let S be a finite set of places of k containing all archimedean places, let n be a positiveintefer, let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n_k$, let $\epsilon > 0$, and let $c \in \mathbb{R}$. Then there is a finite union Z of proper linear subspace of $\mathbb{P}^n_k$, depending only on $k, S, n, H_1, \ldots, H_q, \epsilon$, and c, such that the inequality*

$$\sum_{v \in S} \max_J \sum_{j \in J} \lambda_{H_j,v}(x) \leq (n+1+\epsilon)h(x) + c$$

holds for all $x \in (\mathbb{P}^n_{\mathbb{Q}} \backslash Z)(k)$. Here the set J ranges over all subsets of $\{1, \ldots, q\}$ such that the hyperplanes $(H_j)_{j \in J}$ lie in general position.

We also have the following theorem:

Theorem 1.2 ([Voj23])**.** *Let X be a complete variety over a number field k, let $\mathscr{L}$ be a line sheaf on X with $h^0(X, \mathscr{L}^n) > 1$ for some $N > 0$, and let $\mathbf{D}_1, \ldots, \mathbf{D}_p (p > 0)$ be effective $\mathbb{R}$-Cartier b-divisors on X. Take S be a finite set of places of k. Then, for* *all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zarisiki-closed subset Z of X such that the inequality*

$$m_S(\mathbf{D}_1, \ldots, \mathbf{D}_p, x) \leq (\mathrm{Nev}_{\mathrm{bir}}(\mathscr{L}, \mathbf{D}_1, \ldots, \mathbf{D}_p) + \epsilon) h_{\mathscr{L}}(x) + C$$

holds for all points $x \in X(k) \setminus Z$.

And in [RV20], they give a 'general theorem', and in [Voj23], Vojta generalize the general theorem to the b-divisor case:

Theorem 1.3 ([Voj23]). *Let k be a number field, let X be a complete varitey over k, let $\mathscr{L}$ be a big line sheaf on X, let $p > 0$, and for each $i = 1, \ldots, p$ let $Y_{i,1}, \ldots, Y_{i,q_i}$ be proper closed subschemes of X that have the Autissier property. Let S be a finite set of places of k, and for all i and j and all $v \in S$ let $\lambda_{i,j,v}$ be a Weil function for $Y_{i,j}$ at v. Then, for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequlity*

$$\frac{1}{[k:\mathbb{Q}]} \sum_{v \in S} \max_i \sum_j \beta(\mathscr{L}, Y_{i,j}) \lambda_{Y_{i,j,v}}(x) \leq (1+\epsilon) h_{\mathscr{L}}(x) + C$$

holds for all points $x \in X(k) \backslash Z$.

We omit the analytic part here.

1.2 Subgeneral positions

The subspace theorem has been generalized to many cases, in Evertse and Ferretti [EF08], they generalize the theorem to the arbitrary projective varieties and to divisors which possess a common linear equivalent multiple.

Theorem 1.4 (Evertse-Ferretti,[EF08]). *Let X be a projective variety of dimension n defined over a number field k. Let S be a finite set of places of k. For each $v \in S$, let $D_{0,v}, \ldots, D_{n,v}$ be effective Cartier divisors on X, defined over k, in general position. Suppose that there exists an ample Cartier divisor A on X and positive integers $d_{i,v}$ such that $D_{i,v} \sim d_{i,v} A$ for all i and for all $v \in S$. Let $\epsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subset X$ such that for all points $P \in X(k) \setminus Z$,*

$$\sum_{v \in S} \sum_{i=0}^{n} \frac{\lambda_{D_{i,v},v}(P)}{d_{i,v}} < (n+1+\epsilon) h_A(P).$$

Here, $\lambda_{D_{i,v},v}$ is a local height function associated to the divisor $D_{i,v}$ and place v in S, and h_A is a global (absolute) height associated to A.

Gordon Heier and Aaron Levin further generalized this theorem in their work referenced in [HL21], extending the coefficients from the original work in [EF08] to a concept called the Seshadri constant(Definition 2.15). The specifics are as follows:

Theorem 1.5 (Heier-Levin, [HL21]). *Let X be a projective variety of dimension n defined over a number field k. Let S be a finite set of places of k. For each $v \in S$, let $Y_{0,v}, \ldots, Y_{n,v}$ be closed subschemes of X, defined over k, and in general position. Let A be an ample Cartier divisor on X, and $\epsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subset X$ such that for all points $P \in X(k) \setminus Z$,*

$$\sum_{v \in S} \sum_{i=0}^{n} \epsilon_{Y_{i,v}}(A) \lambda_{Y_{i,v},v}(P) < (n+1+\epsilon) h_A(P).$$

Further, these two authors have extended this theorem in the subgeneral position(Definition 2.14), obtaining the following conclusion:

Theorem 1.6 (Heier-Levin, [HL23]). *Let X be a projective variety of dimension n defined over a number field k, and let S be a finite set of places of k. For each $v \in S$, let $Y_{1,v}, \ldots, Y_{q,v}$ be closed subschemes of X, defined over k (not necessarily in general position), and let $c_{1,v}, \ldots, c_{q,v}$ be nonnegative real numbers. For a closed subset $W \subset X$ and $v \in S$, let*

$$\alpha_v(W) = \sum_{\substack{i \\ W \subseteq \operatorname{Supp} Y_{i,v}}} c_{i,v}.$$

Let A be an ample Cartier divisor on X, and $\epsilon > 0$. Then there exists a proper Zariski closed subset Z of X such that

$$\sum_{v \in S} \sum_{i=1}^{q} c_{i,v} \epsilon_{Y_{i,v}}(A) \lambda_{Y_{i,v},v}(P) < \left((n+1) \max_{\substack{v \in S \\ \emptyset \nsubseteq W \nsubseteq X}} \left(\frac{\alpha_v(W)}{\operatorname{codim} W} \right) + \epsilon \right) h_A(P)$$

for all points $P \in X(k) \setminus Z$.

1.3 Degeneracy of Integral points

Using the approach of subspace theorems, we have proved the following theorem.

Theorem 1.7 ([CZ02]). *Let k be a number field with $\mathcal{O}$ ring of integers, $\tilde{C}$ a projective, absolutely irreducible curve over k, C an affine open subset of $\tilde{C}$, embedded into $\mathbb{A}^m$. Let S be a finite set of places of k. If $\tilde{C}$ has a infinitely many points in $\mathbb{A}^m(\mathcal{O}(S))$, then $\tilde{C}$ has genus 0 and moreover $\#(\tilde{C} \setminus C) \leqslant 2$.*

And we have the following improvements.

Theorem 1.8 ([Lev09], Theorem 10.4A.). *Let X be a nonsingular projective variety defined over a number field k. Let $q = \dim X$. Let $D = \sum_{i=1}^{r} D_i$ be a divisor on X defined over k such that the D_i are effective divisors with no irreducible components in common and such that the intersection of any $m+1$ distinct D_i is empty. Suppose also that every irreducible component of D is nonsingular. If D_i is nef and big for each i and $r > 2[(m+1)/2]q$, then $X \backslash D$ is quasi-Mordellic(See [Lev09], Definition 3.4A.).*

1.4 Main theorem

The following theorem is our main theorem:

Theorem 1.9 (Main theorem)**.** *Let X be a projective variety of dimension n. Let $\mathscr{L}$ be a big line sheaf on X. Let S be a finte set of places. Given a sequence of closed subschemes for each place: $Y_{1,v} \supset Y_{2,v} \supset \cdots \supset Y_{q,v}$ and assume that this is a regular chain. Assume some special conditions and take $\lambda_{Y_{i,v},v}$ to be the correspondance Weil functions, where $v \in S$. Then for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{i=1}^{q} \sum_{v \in s} (\beta(\mathscr{L}, Y_{i,v}) - \beta(\mathscr{L}, Y_{i-1,v}))\lambda_{Y_{i,v},v}(x) \leq (1+\epsilon)h_{\mathscr{L}}(x) + C$$

holds for all points $x \in X(k) \setminus Z$

Use the main theorem, we can give the following theorems:

Theorem 1.10. *Let X be a projective variety of dimension n defined over a number field k. Let $D_1, \ldots, D_{n+1}$ be effective Cartier divisors on X. $a_1, \ldots, a_{n+1}$ are positive integers such that $a_i D_i$ are all numerically equivalent to an ample Cartier divisor D. Let*

$$B = \{\cap_{i \in I \subseteq \{1,\ldots,n+1\}} a_i D_i\}$$

Suppose that $\forall Y_k \in B$, $Y_k \cap a_s D_s = Y_l$, we have

$$\beta(A, Y_l) - \beta(A, Y_k) - \frac{1}{n+1} > 0 \tag{1.1}$$

Let $S \subseteq M_k$ be a finite set of places. $\epsilon > 0$ be any positive number. Then $\exists$ a Zariski closed subset $Z \subseteq X$ such that $\forall$ subset of $(\sum D_i, S)$ integral points $R \subseteq X(k)$, $\forall P \in R \setminus Z$,

$$h_{D_i \cap D_j}(P) \leqslant \epsilon h_D(P)$$

Finally, we mention that via Vojta's dictionary [Voj87, Ch. 3] between Diophantine approximation and Nevanlinna theorey, by substituting Vojta's version [Voj97] of Cartan's Second Main Theorem in place of Schmidt's Subspace Theorem, one can prove a result analogous to Main theorem, giving following inequality in the vein of the Second Main Theorem:

Theorem 1.11. *Let X be a complex projective variety. Let s be a positive integer and let $Y_{1,i} \subset \ldots Y_{q,i}$ be a regular chain of nonempty closed subschemes of X for $i = 1, \ldots, s$. Let A be a big Cartier divisor on X, let $\epsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subset X$ sucht that for all holomorphic maps $f : \mathbb{C} \to X$ whose image is not contained in Z, the inequality:*

$$\int_0^{2\pi} \max_i \left(\sum_{j=1}^{q} (\beta(A, Y_{j,i}) - \beta(A, Y_{j,i}))\lambda_{Y_{j,i}}(f(re^{i\theta})) \right) \frac{d\theta}{2\pi} \leq (1+\epsilon)T_{f,A}(r)$$

holds for all $r \in (0, \infty)$ outside of a set of finite Lebesgue measure.

2 Notations and Preliminaries

Before our discussions, we shoulf first give some notations.

If X is a variety over a number field k, we use $X(M_k)$ denote the disjoint union $\coprod_{v \in M_k} X(\bar{k}_v)$. Where $\bar{k}_v$ means any fixed algebraic closure of k_v

2.1 Weil functions

In this section, we will introduce some basic concepts in the Diophantine approximation.

Definition 2.1. *Let $n \geq 1$, the* multiplicative height *of a rational point $P \in \mathbb{P}^n(\mathbb{Q})$ is defined by*

$$H(P) = \max\{|x_j| : 0 \leq j \leq n\}$$

where we write $P = [x_0 : \cdots : x_n]$ with $(x_0, \cdots, x_n \in \mathbb{Z}^{n+1})$ a primitive integer vector.

Usually, we say height we are concern about logarithmic height, i.e

$$h(P) \coloneqq \log H(P)$$

To generalize this concept, we have the following theorem:

Theorem 2.2 (Weil's height function)**.** *For each irreducible, smooth, projective variety X and invertible sheaf $\mathscr{L}$ on it, both defined over $\mathbb{Q}$, there is a function:*

$$h_{X,\mathscr{L}} : X(\mathbb{Q}) \to \mathbb{R}$$

uniquely defined up to adding a bounded function, such that the following properties hold:

(i) *(Normalization) For $X = \mathbb{P}^n_{\mathbb{Q}}$ and $\mathscr{L} = \mathscr{O}(1)$ er have $h_{\mathbb{P}^1_{\mathbb{Q}},\mathscr{O}(1)}(P) = h(P) + O(1)$*

(ii) *(Functoriality) Given a morphism $f : X \to Y$ of projective varieties over $\mathbb{Q}$ and a line sheaf $\mathscr{L}$ on Y, we have $h_{X,f^*\mathscr{L}}(P) = h_{Y,\mathscr{L}}(f(P)) + O(1)$*

(iii) *(Additivity) Given $\mathscr{L}, \mathscr{M}$ line sheaves on X, we have $h_{X,\mathscr{L}^\vee}(P) = -h_{X,\mathscr{L}}(P) + O(1)$ and $h_{X,\mathscr{L}\otimes\mathscr{M}}(P) = h_{X,\mathscr{L}}(P) + h_{X,\mathscr{M}}(P) + O(1)$*

(iv) *(Isomorphism) if $\mathscr{L} \simeq \mathscr{M}$ on X, then $h_{X,\mathscr{L}}(P) = h_{X,\mathscr{M}}(P) + O(1)$*

(v) *(Effective positivity) If $\mathscr{L}$ is an effective line sheaf on X, then there is $c > 0$ such that $h_{X,\mathscr{L}}(P) \geq c$ for all $P \in X(\mathbb{Q})$ outside the base locus of $\mathscr{L}$*

(vi) *(Ample finiteness) If $\mathscr{A}$ is an ample line sheaf on X, then for each $B > 0$ the set $\{P \in X(\mathbb{Q}) : h_{X,\mathscr{A}}(P) \leq B\}$ is finite*

(vii) *(Numerical equivalence) Let $\mathscr{A}$ and $\mathscr{L}$ be line sheaves on X with $\mathscr{A}$ ample and $\mathscr{L}$ numerically trivial(i.e $\mathscr{L} \equiv \mathscr{O}_X$). Let $\epsilon > 0$. Then for all but finitely many $P \in X(\mathbb{Q})$ we have $|h_{X,\mathscr{L}}(P)| < \epsilon \cdot h_{X,\mathscr{A}}(P)$*

You can view the above definitions in the global cases, now we will work on the local field. For our case, we will only treat the number field k, and we define $M_{\mathbb{Q}}$ is the set of places on k. For simplicity, one can just treat the case $k = \mathbb{Q}_v$.

Definition 2.3 (Weil function)**.** *For each smooth, projective, irreducible variety X over $k = \mathbb{Q}_v$ and any $D \in \mathrm{Div}_k(X)$, a* Weil function *for D is a function*

$$\lambda_{X,v}(D,-) : X(k) - \mathrm{supp}(D) \to \mathbb{R}$$

with the following property: For every $x \in X(k)$ and every $f \in k(X)^{\times}$ local equation for D near X, there is a v-adic neighborhood $W \subset V$ of x and a bounded continuous function $\alpha : W \to \mathbb{R}$ such that for all $P \in W - \mathrm{supp}(D)$ we have

$$\lambda_{X,v} = -\log |f(P)|_v + \alpha(P)$$

Finally, we give our definiton of (D,S)-integral:

Definition 2.4. *Let X be a smooth, irreducible, projective variety over $\mathbb{Q}$. Let $D \in \mathrm{Div}_{\mathbb{Q}}$ be an effective divisor. Suppose that $U = X - \mathrm{supp}(D)$. We say that a set of rational points $\sigma \subset U(\mathbb{Q})$ is (D,S)-integral if for all $P \in \sigma$ we have*

$$\sum_{v \in S} \lambda_{X,v}(D,P) = h_{X,\mathscr{O}(D)}(P) + O(1)$$

In [Sil87], Silverman generalized the Weil height machine for Cartier divisors to height functions on projective varieties with respect to closed subschemes. More precisely, let X be a projective variety over a number field K, and let $Z(X)$ denote the set of closed subschemes of X. Let M_K be the set of places of K. Note that the closed subschemes $Y \in Z(X)$ are in one-to-one correspondence with quasi-coherent ideal sheaves $\mathcal{I}_Y \subseteq \mathcal{O}_X$, and we identify a closed subscheme Y with its ideal sheaf $\mathcal{I}_Y$. Generalizing the Weil height machine for Cartier divisors, Silverman assigned to each $Y \in Z(X)$ and each place $v \in M_K$ a local height function $\lambda_{Y,v}$, and to each $Y \in Z(X)$ a global height function $h_Y = \sum_{v \in M_K} \lambda_{Y,v}$ (both uniquely determined up to a bounded function). We now summarize some of the basic properties of height functions associated to closed subschemes.

Theorem 2.5. *([Sil87]) Let X be a projective variety over a number field K. Let $Z(X)$ be the set of closed subschemes of X. There are maps*

$$\begin{aligned} Z(X) \times M_K &\to \{\textit{functions } X(K) \to [0,+\infty]\}, \\ (Y,v) &\mapsto \lambda_{Y,v}, \\ Z(X) &\to \{\textit{functions } X(K) \to [0,+\infty]\}, \\ Y &\mapsto h_Y, \end{aligned}$$

satisfying the following properties (we also write $\lambda_{X,Y,v}$ and $h_{X,Y}$ for clarity in (6)):

1. *If $D \in Z(X)$ is an effective Cartier divisor, then $\lambda_{D,v}$ and h_D agree with the classical height functions associated to D.*

2. *If $W, Y \in Z(X)$ satisfy $W \subseteq Y$, then $h_W \leq h_Y + O(1)$ and $\lambda_{W,v} \leq \lambda_{Y,v} + O(1)$ for all $v \in M_K$.*

3. *If $W, Y \in Z(X)$ satisfy* $\mathrm{Supp}(W) \subseteq \mathrm{Supp}(Y)$, *then there exists a constant C such that $h_W \leq C \cdot h_Y + O(1)$ and $\lambda_{W,v} \leq C \cdot \lambda_{Y,v} + O(1)$ for all $v \in M_K$.*

4. *For all $W, Y \in Z(X)$, $\lambda_{W\cap Y,v} = \min\{\lambda_{W,v}, \lambda_{Y,v}\} + O(1)$.*

5. *For all $W, Y \in Z(X)$, we have $h_{W+Y} = h_W + h_Y + O(1)$ and $\lambda_{W+Y,v} = \lambda_{W,v} + \lambda_{Y,v} + O(1)$ for all $v \in M_K$.*

6. *Let $\phi : X' \to X$ be a morphism of projective varieties over K, and let $Y \in Z(X)$. Then*

$$h_{X',\phi^*Y} = h_{X,Y} \circ \phi + O(1),$$
$$\lambda_{X',\phi^*Y,v} = \lambda_{X,Y,v} \circ \phi + O(1),$$

for all $v \in M_K$.

7. *If D and E are numerically equivalent Cartier divisors on X and A is an ample divisor on X, then for any $\varepsilon > 0$, we have*

$$|h_D(P) - h_E(P)| < \varepsilon h_A(P) + O(1)$$

for all $P \in X(K)$.

Here, $Y \subset Z$, $Y + Z$, and ϕ^*Y are all defined in terms of the associated ideal sheaves (see [Sil87]). For a closed subscheme Y and finite set of places of S of K, we let $m_{Y,S}(P) = \sum_{v\in S} \lambda_{Y,v}(P)$. For Cartier divisors D and E on a variety X, we will also write $D \geq E$ (or $E \leq D$) if $D - E$ is an effective divisor.

2.2 Birational Part

We will use languages of birational divisors, here are some basic definitions and properties. Here we introduce the notaitons in the [Voj23], anyone interested in this topic can find the explicit proof in it.

we follow the notion of b-divisor given by Shokurov; see [Cor07], where the 'b' stands for the *birational*.

Definition 2.6. *Let X be a complete variety overa field k.*

(a) *A* model *of X is a proper birational morphism $Y \to X$ over k, where Y is a variety over k. We often use Y to denote the model.*

(b) *The category of models of X is the category whose objects are models of X and whose morphisms are morphism over X. We say that a model Y_1 of X* dominates *a model Y_2 of X if there is a morphism $Y_1 \to Y_2$(necessarily unique) in this category.*

(c) *A* b-Cartier divisor *(resp. $\mathbb{Q}$-b-Cartier divisor) on X is an equivalence class of pairs (Y, D), where Y is a model of X and D is a Cartier divisor (resp. $\mathbb{Q}$-Cartier divisor) on Y; here equivalence classes are those for the equivalence relation generated by the relation $(Y_1, D_1) \sim (Y_2, D_2)$ if Y_1 dominates Y_2 via $\phi : Y_1 \to Y_2$ and $D_1 = \phi^* D_2$.*

(d) *A b-Cartier divisor or $\mathbb{Q}$-b-Cartier divisor $\mathbf{D}$ on X is* effective *if it is represented by a pair (Y, D) such that D is effective.*

Remark 2.7. *Here is some basic remarks on the definition above:*

1. *Here we can also given the definition more directly, i.e we can define:*

$$\mathbf{D} = (\mathbf{D}_Y)Y \in \varprojlim_{Y} \mathrm{Div}(Y)$$

   when the X is required to be normal. In this case, one can check that the two definitons conincide, at the same time, we will always work on the case that X is normal.

2. *The definition for the effective is well-defined, for one can work on the models which is dominate two given models on X, then one can move the effective properties along it.*

3. *For the blow up morphism is always projective, hence proper, we know that any closed subschemes can be lift to a model, in which it is a divisor. Hence any closed subschemes in X can view as one b-divisor, that is one reason why the b-divisors genrealize the usual divisors.*

We have just given the definition of the b-divisors, so it is natural to consider the b-Weil functions:

Definition 2.8. *Let X be a complete variety over a number field k. Then a* b-Weil function *on X (resp. a $\mathbb{Q}$-b-Weil function on X) is an equivalence class of pairs (U, λ), where U is a nonempty Zariski-open subset of X and $\lambda : U(M_k) \to \mathbb{R}$ is a function such that there exist a model $\phi : Y \to X$ of X and a Cartier divisor(resp. $\mathbb{Q}$-b-Cartier divisor) D on Y such that $\lambda \circ \phi$ extend to a Weil function for D(resp. such that $n\lambda \circ \phi$ extends to a Weil function for nD for some(and hence all) nonzero integers n for which nD is a Cartier divisor). Pairs (U, λ) and (U', λ') are* equivalent *if $\lambda = \lambda'$ on $(u \cap U')(M_k)$. Local b-Weil functions and local $\mathbb{Q}$-b-Weil functions on X are defined similarly.*

Definition 2.9. *Let X be a complete variety over a number field k, let λ be a b-Weil function on X, and let $\mathbf{D}$ be a b-cartier divisor on X. We say that λ is a b-Weil function for $\mathbf{D}$ if $\mathbf{D}$ is represented by a pair (Y, D) as above, such that if $\phi : Y \to X$ is the structural morphism of Y, then $\lambda \circ \phi$ extends to a Weil function for D on Y.*

The conncetion between the b-Weil funciton and b-Cartier divisor is just the same as the usual case. Here we just give a list of the properties:

Proposition 2.10. *Let X be a complete variety over a number field k. Given $\mathbf{D}_i$ and λ_i respectively/*

(a) *$-\lambda_i$ correspond to $-\mathbf{D}_i$ and $\lambda_1 + \lambda_2$ correspond to $\mathbf{D}_1 + \mathbf{D}_2$*

(b) *λ_i is M_k bounded iff $\mathbf{D}_i$ is effective.*

(c) *Modulo a M_k constant, we have the fact that the b-Weil function and the b-Cartier funciton are one-to-one corresponding.*

One could define a partial order on the set of b-Cartier divisors, more explicitly, one may define $\mathbf{D}_1 \geq \mathbf{D}_2$ iff $\mathbf{D}_1 - \mathbf{D}2$ is effective. One surprising result is that with the partial order given above, the b-Cartier divisors form a lattice, i.e. it has a least upper bound and the greatest lower bound. As for the partial order is compatible with the group action, we only need give the description of the least lower bound.

Lemma 2.11. *Let X/k as above, let $\mathbf{D}$ and $\mathbf{D}_1, \dots, \mathbf{D}_l$ be b-Cartier divisors on X. Choose a model Y of X such that $\mathbf{D}_i$ is represented D_i where D_i is a traditional Cartier divisor on Y, and $\mathbf{D}$ is represented by D. Then we have the fact that $\mathbf{D}$ is a least upper bound of $\mathbf{D}_1, \dots, \mathbf{D}_l$ if and only each $\mathbf{D} - \mathbf{D}_i$ is effective and we have:*

$$\bigcap_{i=1}^{l} \mathrm{Supp}(D - D_i) = \emptyset$$

This lemma is easy to proof, briefly, one just consider the models which blow up the $D - D_i$, and then one can find a lower upper bound. Hence one must require that $\bigcap_{i=1}^{l}(D - D_i)$ is $\emptyset$.

We also have that if we assume λ_i for the correseponding b-Weil divisor to the $\mathbf{D}_i$, then we have $max\{\lambda_1, \lambda_2\} = \lambda_{\mathbf{D}_1 \vee \mathbf{D}_2}$, where we use the $\mathbf{D}_1 \vee \mathbf{D}_2$ to represent the least upper bound of the $\mathbf{D}_1$ and $\mathbf{D}_2$.

We can define the global section of b-divisor.

Definition 2.12 ([Voj23], Definition 6.3.)**.** *Let $\mathscr{L}$ be a line sheaf on X and let $\mathbf{D}$ be an effective Cartier b-divisor on X. Then*

$$H^0_{bir}(X, \mathscr{L}(-\mathbf{D})) = H^0(W, \pi^*\mathscr{L}(-D))$$

, where $\pi : W \to X$ is any normal model of X on which $\mathbf{D}$ is represented by a Cartier divisor D. Also,

$$h^0_{bir}(X, \mathscr{L}(-\mathbf{D})) = \dim_k H^0_{bir}(X, \mathscr{L}(-\mathbf{D}))$$

.

Lemma 2.13 ([Voj23], Lemma 6.4.)**.** *Let $\mathscr{L}$ be a line sheaf on X, let D be a nonzero effective Cartier divisor on X, and let $d = \dim X$. Then*

$$h^0_{bir}(X, \mathscr{L}^N) = h^0(X, \mathscr{L}^N) + O(N^{d-1})$$

and

$$\sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N(-mD)) = \sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N(-mD)) + O(N^d)$$

.

2.3 Some constants

We will firmly work on this special constant, here we first give the basic definition of it:

Definition 2.14 (m-Subgeneral Position)**.** *Let X be a projective variety of dimension n. We say that closed subschemes $Y_1, \ldots, Y_q$ of X are in m-subgeneral position if for every subset $I \subset \{1, \ldots, q\}$ with $|I| \leq m+1$, we have*

$$\operatorname{codim} \bigcap_{i \in I} Y_i \geq |I| + n - m,$$

where we use the convention that $\dim \emptyset = -1$. In the case $m = n$, we say that the closed subschemes are in general position. If V is a subset of X, we say that closed subschemes $Y_1, \ldots, Y_q$ of X are in general position outside of V if for every subset $I \subset \{1, \ldots, q\}$ with $|I| \leq n+1$ we have $\operatorname{codim}((\bigcap_{i \in I} Y_i) \setminus V) \geq |I|$.

Definition 2.15 (Seshadri constants)**.** *Let Y be a closed subscheme of a projective variety X and let $\pi : \tilde{X} \to X$ be the blowing-up of X along Y. Let A be a nef Cartier divisor on X. We define the Seshadri constant $\epsilon_Y(A)$ of Y with respect to A to be the real number*

$$\epsilon_Y(A) = \sup\{\gamma \in \mathbb{Q}_{\geq 0} \mid \pi^* A - \gamma E \text{ is } \mathbb{Q}\text{-nef}\},$$

where E is an effective Cartier divisor on $\tilde{X}$ whose associated invertible sheaf is the dual of $\pi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_{\tilde{X}}$.

Definition 2.16 (β constant)**.** *Let $\mathscr{L}$ be a big line sheaf on X, let Y be a nonempty proper closed subscheme of X, and let $\mathscr{I}$ be the sheaf of ideals corresponding to Y. Then*

$$\beta(\mathscr{L}, Y) = \liminf_{N \to \infty} \frac{\sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N \otimes \mathscr{I}^m)}{N h^0(X, \mathscr{L}^N)}$$

Since we have talked about so much with the b-divisor, it is a natural question that can we generalize the concept of the β constant into some 'b-divisor' case. Use the definition above, we have the following defintion:

Definition 2.17.

$$\beta(\mathscr{L}, \mathbf{D}) = \liminf_{N \to \infty} \frac{\sum_{m=1}^{\infty} h^0_{bir}(X, \mathscr{L}^N(-m\mathbf{D}))}{N h^0_{bir}(X, \mathscr{L}^N)}$$

. This definition is well-defined. Acoording to the lemma above, we know when we take different representative element for the b-divisor $\mathbf{D}$, the variation of the numerator of the β constant is at most $O(N^d)$. And one reason that we define above is the following proposition:

Proposition 2.18 ([Voj23], Corollary 6.9.)**.** *Let $\mathbf{Y}$ be the b-divisor corresponding to a proper closed subscheme of X. Let $\mathscr{L}$ be a big line sheaf on X. Then:*

$$\beta(\mathscr{L}, Y) = \beta(\mathscr{L}, \mathbf{Y})$$

We give another constant, the Nevanlinna constant:

Definition 2.19 (Nevanlinna constant)**.** *Let X be a complete variety, let D be an effective Cartier divisor on X, and let $\mathscr{L}$ be a line sheaf on X. If X is normal, then we define*

$$\mathrm{Nev}_{bir}(\mathscr{L}, D) = \inf_{N,V,\mu} \frac{\dim V}{\mu}$$

where the infimun passes over all triples (N, V, μ) such that $N \in \mathbb{Z}_{>0}$, V is a linear subspace of $H^0(X, \mathscr{L}^N)$ with $\dim V > 1$, and $\mu \in \mathbb{Q}_{>0}$, with the following property. There exist a variety Y and a proper birational morphism $\phi : Y \to X$ such that the following condition holds. For all $Q \in Y$ there is a basis $\mathcal{B}$ of V such that

$$\phi^*(\mathcal{B}) \geq \mu N \phi^* D$$

in a Zariski-open neighborhood U of Q, relative to the cone of effective $\mathbb{Q}$-divisors on U. If there are no such triples (N, V, μ), then Nev_{bir} is defined to be $+\infty$. For a general complete variety X, $\mathrm{Nev}_{bir}(\mathscr{L}, D)$ is defined by pulling back to the normalization of X.

3 Filtration method

Filtraion method has been widely used in Diophantine approximation. In our work, we will also use this techenice, hence here we give a brief review of it.

3.1 The closed subscheme cases

We will give our definition for the intersect properly. When our base ringis Cohen-Macauley, we know that the concept of intersect peoperly and in general position defined above are the same.

Definition 3.1. *Let $I_1, \ldots, I_q$ be ideals of A, with $q \in \mathbb{N}$. Then $I_1, \ldots, I_q$ intersect properly if*

(i) *for each $i = 1, \ldots, q$ there is a nonempty regular sequence $\phi_{i1}, \ldots, \phi_{ir_i}$ in A such that I_i is of monomial type with repect to $\phi_{i1}, \ldots, \phi_{ir_i}$, i.e. $I = \mathscr{J}(N)$ with respect to $\phi_{i1}, \ldots, \phi_{ir_i}$ and some saturated N.*

(ii) *the sequence $\phi_{11}, \ldots, \phi_{1r_1}, \ldots, \phi_{q1}, \ldots, \phi_{qr_q}$ is a reugular sequence.*

Back to the scheme cases, as in the [Voj23], we have the following definition:

Definition 3.2 ([Voj23], Definition 4.1)**.** *Let $\mathscr{J}_1, \ldots, \mathscr{J}_q$ be the ideal sheaves that corresponding to $Y_1, \ldots, Y_q$, where Y_i are proper closed subscheme of X, a complete variety.*

(a) *We say that $Y_1, \ldots, Y_q$ intersect properly at a point $P \in X$ if the subsequence of proper ideals in the sequence $(\mathscr{J}_1)_P, \ldots, (\mathscr{J}_q)_P$ of ideals of the local ring $\mathscr{O}_{X,P}$ intersect properly. If $P \notin \cup Y_i$, this is naturally correct.*

(b) *We say that $Y_1, \ldots, Y_q$ intersect properly if $Y_1, \ldots, Y_q$ intersect properly at all points of X.*

(c) *We say that $Y_1, \ldots, Y_q$ weakly intersect properly if they intersect properly at all $P \in \bigcup_{i \neq j}(Y_i \cap Y_j)$*

3.2 Filtration Method

In this section, we will know why should have the Autissier property in our works. We will introduce a poweful method in our job, called the filtration method. Its basic is the following.

We first give the most basic type of the filtration method:

Lemma 3.3 ([Lev09], Lemma 10.1.)**.** *Let V be a vector space of finite dimension d over a field k. Let $V = W_1 \supset W_2 \supset \cdots \supset W_h$ and $V = W_1^* \supset W_2^* \supset \cdots \supset W_{h^*}^*$ be two filtrations on V. There is a basis $v_1, \ldots, v_d$ of V that contains a basis of each W_j and W_j^*.*

Definition 3.4. *Let $r \in \mathbb{Z}_{>0}$. A subset N of $\mathbb{N}^r$ is saturated if it is nonempty and if $N \supset \mathbf{a} + \mathbb{N}^r$ for all $\mathbf{a} \in N$.*

Definition 3.5. *Let $\phi_1, \ldots, \phi_r \in A$ with $r > 0$, and let N be a saturated subset of $\mathbb{N}^r$. Then $\mathscr{I}(N)$ is the ideal of A of generated by the set $\{\phi_1^{b_1}, \ldots \phi_r^{b_r} : \mathbf{b} \in N\}$.*

In the [Aut11], Autissier find the following key lemma:

Lemma 3.6. *Let $\phi_1, \ldots, \phi_r (r > 0)$ be a regular sequence in A, and let N_1 and N_2 be saturated subsets of $\mathbb{N}^r$. Then*

$$\mathscr{I}(N_1 \cap N_2) = \mathscr{I}(N_1) \cap \mathscr{I}(N_2)$$

For our purpose, we will give a bit generalization of this concept into the ideal case.

Definition 3.7. *Let I be an ideal of A and let $\phi_1, \ldots, \phi_r$ be a sequence of elements of A. Then I is of monomial type with respect to $\phi_1, \ldots, \phi_r$ if $r > 0$ and $I == \mathscr{I}(N)$ (taken relative to $\phi_1, \ldots, \phi_r$) for some saturated subset N of $\mathbb{N}^r$.*

Here, we define the n-multiple of the saturated set as below:

Definition 3.8. *When n=0, we define $0N = \mathbb{N}^r$, when $N > 0$, we define*

$$nN \coloneqq \{\mathbf{b}_1 + \cdots + \mathbf{b}_n : \mathbf{b}_1, \ldots, \mathbf{b}_n \in N\}$$

We now replace the divisors into closed subschemes, we have the following definiton:

Definition 3.9. *Let $q \in \mathbb{Z}_{>0}$, let $I_1, \ldots, I_q$ be ideals in A, let N be a saturated subset of N^q. Then $\mathscr{I}(N)$ is the ideal of A defined by*

$$\mathscr{I}(N) = \sum_{\mathbf{b} \in N} I_1^{b_1} \cdots I_q^{b_q}$$

Hence, we define the Autissier property as below:

Definition 3.10. *Let $I_1, \ldots, I_q$ be ideals in A. We say that they have the Autissier peoperty if*

$$\mathscr{J}(N \cap N') = \mathscr{J}(N) \cap \mathscr{J}(N')$$

Definition 3.11 ([Voj23], Definition 4.2)**.** *Let $\mathscr{J}_1, \ldots, \mathscr{J}_q$ be as in 3.2*

(a) *Let $P \in X$, and let $j_1, \ldots, j_r$ be the subsequence of $1, \ldots, q$ consisting of those j such that $P \in Y_j$. We say that $Y_1, \ldots, Y_q$ have the Autissier property at P if*

$$\mathscr{J}(N \cap N') = \mathscr{J}(N) \cap \mathscr{J}(N')$$

for all saturated subsets N and N' of $\mathbb{N}^r$, where $\mathscr{J}$ is taken with respect to the sequence $(\mathscr{J}_{j1})_p, \ldots, (\mathscr{J}_{jr})_P$ of proper ideals of $\mathscr{O}_{X,P}$.

(b) *We say that $Y_1, \ldots, Y_q$ have the Autissier property if they have the Autissier property at all $P \in X$.*

In the [Voj23], Vojta deeply study the Autissier property, and given the following useful propersition:

Proposition 3.12. *If $Y_1, \ldots, Y_q$ weakly intersect properly, then they have the Autissier properly.*

Definition 3.13. *Let W be a vector space of finite dimension. A filration of W is a family of subspaces $\mathcal{F} = (\mathcal{F}_x)_{x \in \mathbb{R}^+}$ of subspaces of W such that $\mathcal{F}_x \supset \mathcal{F}_y$ whenever $x \leq y$, and such that $\mathcal{F}_x = 0$ for x big enough. A basis $\mathcal{B}$ of W is said to be adapted to $\mathcal{F}$ if $\mathcal{B} \cap \mathcal{F}_x$ is a basis of $\mathcal{F}_x$ for every real number $x \geq 0$.*

Lemma 3.14 (Corvaja-Zannier[CZ04],Levin[Lev09],Autissier[Aut11])**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be two filtrations of W. Then there exists a basis of W which is adapted to both $\mathcal{F}$ and $\mathcal{G}$.*

Turning to the consequence of the Autissier property, we need the following setting.

Proposition 3.15 ([Aut11],[RV20])**.** *Let $q \in \mathbb{Z}_{>0}$, let*

$$\square = \mathbb{R}^q_{\geq 0} \backslash \{\mathbf{0}\}$$

and for all $\mathbf{t} in \square$ and all $x \in \mathbb{R}_{\geq 0}$ let

$$N(\mathbf{t}, x) = \{\mathbf{b} \in \mathbb{N}^q : t_1 b_1 + \cdots + t_q b_q \geq x\}.$$

Let $I_1, \ldots, I_q$ be ideals in A that have the Autissier property. Then

$$\mathscr{J}(N(\mathbf{t}, x)) \cap \mathscr{J}(N(\mathbf{u}, y)) \subset \mathscr{J}(N(\lambda\mathbf{t} + (1-\lambda)\mathbf{u}, \lambda x + (1-\lambda)y))$$

for all $\mathbf{t}, \mathbf{u} \in \square$, all $x, y \in \mathbb{R}_{\geq 0}$, and all $\lambda \in [0, 1]$

Definition 3.16 ([Voj23]). *Let* $\square$ *and* $N(\mathbf{t},x)$ *be as above, fix a complete variety* X *over a field* k *and proper closed subschemes* $Y_1,\dots,Y_q$ *of* X. *Let* $\mathscr{I}_1,\dots,\mathscr{I}_q$ *be the corresponding ideal sheaf.*

(a) *Let* N *be a saturated subset of* $\mathbb{N}^q$. *Then*

$$\mathscr{I}_X(N)=\sum_{\mathbf{b}\in N}\mathscr{I}_1^{b_1}\cdots\mathscr{I}_q^{b_q}$$

This is a coherent ideal sheaf in $\mathscr{O}_X$

(b) *For each* $\mathbf{t}\in\square$ *and all* $x\in\mathbb{R}_{\geq 0}$, *let*

$$\mathscr{I}_X(\mathbf{t},x)=\mathscr{I}_X(N(\mathbf{t},x))=\sum_{b\in N(\mathbf{t},x)}\mathscr{I}_1^{b_1}\cdots\mathscr{I}_q^{b_q}$$

(c) *Fix a line sheaf* $\mathscr{L}$ *on* X, *and let* $\mathbf{t}$ *and* x *be as above. Then we let*

$$\mathscr{F}(\mathbf{t})_x=\mathscr{F}_{\mathscr{L}}(\mathbf{t})_x=H^0(X,\mathscr{L}\otimes\mathscr{I}_X(\mathbf{t},x))$$

Then $(\mathscr{F}(\mathbf{t})_x)_{x\in\mathbb{R}_{\geq 0}}$ *is a descending filtration of* $H^0(X,\mathscr{L})$ *that satisfies* $\mathscr{F}(\mathbf{t})_x=0$ *for all* $x\gg 0$.

(d) *Finally, for all* $\mathbf{t}\in\square$ *we let*

$$F(\mathbf{t})=F_{\mathscr{L}}(\mathbf{t})=\frac{1}{h^0(X,\mathscr{L})}\int_0^\infty(\dim\mathscr{F}(\mathbf{t})_x)dx.$$

Here the function we defined $F(\mathbf{t})$ was widely used in the proof of the 1.3. And will be used in our situations.

4 Main Theorem

In this section, we will give our approach of proving the main theorem, Here we repeat the main theorem again:

Theorem 4.1 (Main theorem). *Let* X *be a projective variety of dimension* n. *Let* $\mathscr{L}$ *be a big line sheaf on* X. *Let* S *be a finte set of places. Given a sequence of closed subschemes:* $Y_{1,v}\supset Y_{2,v}\supset\cdots\subset Y_{q,v}$ *for each place* $v\in S$ *and assume that this is a regular chain. Assume some special conditions and take* $\lambda_{Y_{i,v},v}$ *to be the correspondance Weil functions, where* $v\in S$. *Then for all* $\epsilon>0$ *and all* $C\in\mathbb{R}$, *there is a proper Zariski-closed subset* Z *of* X *such that the inequality*

$$\sum_{i=1}^{q}\sum_{v\in s}(\beta(\mathscr{L},Y_{i,v})-\beta(\mathscr{L},Y_{i-1,v}))\lambda_{Y_{i,v},v}(x)\leq(1+\epsilon)h_{\mathscr{L}}(x)+C$$

holds for all points $x\in X(k)\setminus Z$

We have two ways to approach the proof of the main theorem, we will represent them in order.

4.1 Filtration method

In this section, we will follow the ideal of traditional method and in some case prove our theorem. First, we call a function is pseudo concave if:

Definition 4.2. *Given a positive-valued function $F(t)$, we call $F(t)$ is pseudo concave if for any $\mathbf{t} = (t_1, \ldots, t_n)$ with $t_i \geq 0$, $(\beta_1, \ldots, \beta_n)$ with $\beta_i \geq 0$ and the property: $\sum_{i=1}^{n} \beta_i t_i = 1$, we have:*

$$F(\mathbf{t}) \geq \min\{\frac{1}{\beta_i}(F(\mathbf{e}_i) - F(\mathbf{e}_{i-1}))\}$$

Now we will give our first approach in proving the main theorem:

Theorem 4.3 (Main theorem: Filtration version)**.** *Let X be a projective variety of dimension n. Let $\mathscr{L}$ be a big line sheaf on X. Let S be a finte set of places. Given a sequence of closed subschemes: $Y_{1,v} \supset Y_{2,v} \supset \cdots \supset Y_{q,v}$ for each $v \in S$ and assume that this is a regular chain. Assume the function $F(t)$ in 3.16 is pseudo concave. Take $\lambda_{Y_{i,v},v}$ to be the correspondance Weil functions, where $v \in S$. Then for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{i=1}^{q} \sum_{v \in s} (\beta(\mathscr{L}, Y_{i,v}) - \beta(\mathscr{L}, Y_{i-1,v}))\lambda_{Y_{i,v}}(x) \leq (1+\epsilon) h_{\mathscr{L}}(x) + C$$

holds for all points $x \in X(k) \backslash Z$

Proof. In this case, we consider that all the $Y_{i,v}$ comes from a explict Y_i. The general case is almost the same. See the arguement in [RV20]. We denote by $m_s(x, Y_i) = \sum_{v \in S} \lambda_{Y_i,v}(x)$. For we have the fact that $m_S(x, Y_i)/h_{\mathscr{L}}(x)$ is bounded when x lies outside of a proper Zariski-closed subset, it suffices to prove the assumption with a bit smaller ϵ and β, actually, it means that we can replace ϵ and $\beta(\mathscr{L}, Y_i)$ to be rational numbers close to them.

Choose positvie intergers N and b such that

$$(1 + \frac{n}{b}) \max_{1 \leq i \leq q} \frac{(\beta_i - \beta_{i-1}) N h^0(X, \mathscr{L}^N)}{\sum_{m \geq 1} (h^0(X, \mathscr{L}^N(-mY_i)) - h^0(X, \mathscr{L}^N(-mY_{i-1})))} < 1 + \epsilon$$

Let

$$\Sigma = \{\sigma \subset \{1, \ldots, q\}\}$$

be a index set.

For $\sigma \, in \Sigma$, let

$$\triangle_\sigma = \{\mathbf{a} = (a_i) \in \prod_{i \in \sigma} (\beta_i - \beta_{i-1})^{-1} \mathbb{N} \mid \sum_{i \in \sigma} (\beta_i - \beta_{i-1}) a_i = b\}$$

For $\mathbf{a} \in \triangle_\sigma$ as above, we defines the ideal $\mathscr{I}_{\mathbf{a}}(x)$ by

$$\mathscr{I}_{\mathbf{a}}(x) = \sum_{\mathbf{b}} \mathscr{I}_{Y_1}^{b_1} \cdots \mathscr{I}_{Y_q}^{b_q}$$

where the sum is taken for all $\mathbf{b} \in \mathbb{N}^{\#\sigma}$ with $\sum_{i\in\sigma} a_i b_i \leq X$. Let

$$\mathscr{F}(\sigma;\mathbf{a})_x = H^0(X, \mathscr{L}^N \otimes \mathscr{J}_{\mathbf{a}}(x))$$

which we regard as a subspace of $H^0(X, \mathscr{L}^N)$, and let

$$F(\sigma, \mathbf{a}) = \frac{1}{h^0(\mathscr{L}^N)} \int_0^{+\infty} (\dim \mathscr{F}(\sigma;\mathbf{a})_x) dx$$

Use the pseudo concave, we have the fact that

$$F(\sigma, \mathbf{a}) \geq \min_{1\leq i\leq q} \left(\frac{b}{\beta_i - \beta_{i-1}} \frac{\sum_{m\geq 1}(h^0(\mathscr{L}^N(-mY_i)) - h^0(\mathscr{L}^N(-mY_{i-1})))}{h^0(\mathscr{L}^N)} \right)$$

We also define

$$\mu_{\mathbf{a}}(s) = \sup\left\{x \in \mathbb{R}^+ : s \in \mathscr{F}(\sigma,\mathbf{a})_x\right\}$$

Let $\mathcal{B}_{\sigma,\mathbf{a}}$ be a basis of $H^0(X, \mathscr{L}^N)$ adapted to the filtration above, then we have $F(\sigma, \mathbf{a}) = \frac{1}{h^0(\mathscr{L}^N)} \sum_{s\in\mathcal{B}_{\sigma,\mathbf{a}}} \mu_{\mathbf{a}}(s)$. Hence

$$\sum_{s\in\mathcal{B}_{\sigma,\mathbf{a}}} \mu_{\mathbf{a}}(s) \geq \min_{1\leq i\leq q} \frac{b}{\beta_i - \beta_{i-1}} \sum_{m\geq 1}(h^0(\mathscr{L}^N(-mY_i)) - h^0(\mathscr{L}^N(-mY_{i-1})))$$

One important fact is that there are only finite many $(\sigma, \mathbf{a})$ with $\sigma \in \Sigma$ and $\mathbf{a} \in \triangle_\sigma$.

Let $\sigma \in \Sigma$, $\mathbf{a} \in \triangle_\sigma$ and $s \in H^0(X, \mathscr{L}^N)$ with $s \coloneqq 0$. One can show that the superemum is actually a maximum.

Similarly, we have

$$\mathscr{L}^N \otimes \mathscr{J}_{\mathbf{a}}(\mu_{\mathbf{a}}(s)) = \sum_{\mathbf{b}\in K} \mathscr{L}^N(-\sum_{i\in\sigma} b_i Y_i)$$

Where $K = K_{\sigma,\mathbf{a},s}$ is the set of minimal elements of $\left\{\mathbf{b} \in \mathbb{N}^{\#\sigma} \mid \sum_{i\in\sigma} a_i b_i \leq \mu_{\mathbf{a}}(s)\right\}$ relative to the product partial ordering on $\mathbb{N}^{\#\sigma}$. This set is finite, so by [RV20] proposition 4.18, we have:

$$(s) \geq \bigwedge_{\mathbf{b}\in K} \sum_{i\in\sigma} b_i Y_i$$

For a basis $\mathcal{B}$ of $H^0(X, \mathscr{L}^N)$, we denote $(\mathcal{B})$ the sum of the divisors (s) for all $s \in \mathcal{B}$

Lemma 4.4.

$$\bigvee_{\substack{\sigma\in\Sigma \\ \mathbf{a}\in\triangle_\sigma}} (\mathcal{B}_{\sigma,\mathbf{a}}) \geq \frac{b}{b+n} \left(\min_{1\leq i\leq q} \sum_{m=1}^{\infty} \frac{h^0(X, \mathscr{L}^N(-mY_i)) - h^0(X, \mathscr{L}^N(-mY_{i-1}))}{\beta_i - \beta_{i-1}} \right) \sum_{i=1}^{q} (\beta_i - \beta_{i-1}) Y_i$$

Proof. Let $\mathbf{D}' = \bigwedge_{\sigma,\mathbf{a}} (\mathcal{B}_{\sigma,\mathbf{a}})$, let $\phi : Y \to X$ be a normal model of X on which $\mathbf{D}', Y_i$ are represented by Cartier divisors D', D_i, let E be a prime divisor on Y. For some point $P \in \phi(\mathrm{Supp} E)$, let

$$\sigma = \{i \in \{1, \ldots, q\} : P \in \mathrm{Supp} Y_i\}$$

Let $\nu', \nu_{\sigma,\mathbf{a}}$ and ν_i be the multipicity of E in $D', \phi^*(\mathcal{B}_{\sigma,\mathbf{a}})$ and D_i repsectively. Let $\nu = \sum_{i=1}^{q}(\beta_i - \beta_{i-1})\nu_i$. Since $\nu' \geq \nu_{\sigma,\mathbf{a}}$ for all $\mathbf{a} \in \triangle_\sigma$, the proof is equivalent to show there is a $\mathbf{a}$ such that

$$\nu_{\sigma,\mathbf{a}} \geq \frac{b}{b+n}\left(\min_{1\leq i\leq q}\sum_{m=1}^{\infty}\frac{h^0(X,\mathscr{L}^N(-mY_i)) - h^0(X,\mathscr{L}^N(-mY_{i-1}))}{\beta_i - \beta_{i-1}}\right)\nu$$

The case that $\nu = 0$ is nothing to prove, hence we can assume that $\nu > 0$

For $i \in \sigma$,let

$$t_i = \frac{v_i}{v}$$

Then we have

$$\sum_{i\in\sigma}(\beta_i - \beta_{i-1})\nu_i = \sum_{i=1}^{q}(\beta_i - \beta_{i-1})\nu_i = \nu$$

Hence we have the fact that $\sum(\beta_i - \beta_{i-1})t_i = 1$. Therefore $b \leq \sum_{i\in\sigma}\lfloor(b+n)(\beta_i - \beta_{i-1})t_i\rfloor \leq b+n$. So we may choose $\mathbf{a} = (a_i) \in \triangle_\sigma$ such that

$$a_i \leq (b+n)t_i$$

For any $s \in \mathcal{B}_{\sigma,\mathbf{a}}$ let ν_s be the multiplicity of E in the divisor $\phi^*(s)$. Hence we have:

$$\mu_s \geq \min_{\mathbf{b}\in K}\sum_{i\in\sigma} b_i\nu_i = \left(\min_{\mathbf{b}\in K}\sum_{i\in\sigma} b_i t_i\right)\nu \geq \left(\min_{\mathbf{b}\in K}\sum_{i\in\sigma}\frac{a_i b_i}{b+n}\right)\nu \geq \frac{\mu_{\mathbf{a}}(s)\nu}{b+n}$$

Hence we have

$$\frac{\nu_{\sigma,\mathbf{a}}}{\nu} = \frac{1}{\nu}\sum_{s\in\mathcal{B}_{\sigma,\mathbf{a}}}\nu_s \geq \frac{1}{b+n}\sum_{s\in\mathcal{B}_{\sigma,\mathbf{a}}}\mu_{\mathbf{a}}(s) \geq \frac{b}{b+n}\left(\min_{1\leq i\leq q}\sum_{m=1}^{\infty}\frac{h^0(X,\mathscr{L}^N(-mY_i)) - h^0(X,\mathscr{L}^N(-mY_{i-1}))}{\beta_i - \beta_{i-1}}\right)$$

□

By the lemma above, actually we find a triple (N, V, μ) satisfied the definition of the Nevanlinna constant, with

$$\frac{\dim V}{\mu} = \left(1 + \frac{n}{b}\right)\max_{1\leq i\leq q}\frac{(\beta_i - \beta_{i-1})Nh^0(X,\mathscr{L}^N)}{\sum_{m\geq 1}h^0(X,\mathscr{L}^N(-mY_i))} < 1 + \epsilon$$

Hence we have

$$\mathrm{Nev}_{bir}\left(\mathscr{L}, \sum_{i=1}^{q}(\beta_i - \beta_{i-1})Y_i\right) \leq 1$$

Hence, by the 1.2, we know that the main theorem holds in this case. □

4.2 Birational method

Definition 4.5. *Let X be a projective variety over a number field k, $X \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots \supseteq Y_q$ be a chain of closed subschemes of X. We say this chain* ***blowup appropriately****, if:*

We blow up Y_i inductively(Let $X_1 \xrightarrow{Bl_{Y_1}} X_0 = X$ be the blowup of Y_1. Then define $Y_2' = Bl_{Y_1}^* Y_2$, therefore we get $X_2 \xrightarrow{Bl_{Y_2}'} X_1$. Repeat this progress.), then we get a chain of birational morphisms:

$$X_q \xrightarrow{Bl_{Y_q'}} X_{q-1} \xrightarrow{Bl_{Y_{q-1}'}} \ldots \xrightarrow{Bl_{Y_2'}} X_1 \xrightarrow{Bl_{Y_1'}} X_0 = X$$

Let D_i(on X_q) be the strict transform of the exceptional divisor(on X_i) of Y_i' and denote the composite of all the maps above by $\pi : X_q \to X$. Then $\pi^*(Y_i) = \sum_{j=i}^{r} \tilde{D}_j$. The condition is $\tilde{D}_1, \ldots, \tilde{D}_r$ have the Autissier property.

Assume this condition, we get a new version of our main theorem.

Theorem 4.6 (Main theorem: birational version)**.** *Let X be a projective variety of dimension n. Let $\mathscr{L}$ be a big line sheaf on X. Let S be a finte set of places. Given a sequence of closed subschemes: $Y_{1,v} \supset Y_{2,v} \supset \cdots \supset Y_{q,v}$ for each $v \in S$ and assume that this chain blowup appropriately. Take $\lambda_{Y_{i,v},v}$ to be the correspondance Weil functions, where $v \in S$. Then for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{i=1}^{q} \sum_{v \in s} (\beta(\mathscr{L}, Y_{i,v}) - \beta(\mathscr{L}, Y_{i-1,v}))\lambda_{Y_{i,v}}(x) \leq (1+\epsilon)h_{\mathscr{L}}(x) + C$$

holds for all points $x \in X(k) \backslash Z$

Proof.

$$\begin{aligned}
&\sum_{i=1}^{q} \sum_{v \in s} (\beta(\mathscr{L}, Y_{i,v}) - \beta(\mathscr{L}, Y_{i-1,v}))\lambda_{Y_{i,v}}(x) \\
=&\sum_{v \in S} \sum_{i=1}^{q} \beta(\mathscr{L}, Y_{i,v})(\lambda_{Y_{i,v}}(x) - \lambda_{Y_{i+1,v}}(x)) \\
=&\sum_{v \in S} \sum_{i=1}^{q} \beta(\mathscr{L}, \mathbf{Y}_{i,v})(\lambda_{\mathbf{Y}_{i,v}}(x) - \lambda_{\mathbf{Y}_{i+1,v}}(x)) \\
=&\sum_{v \in S} \sum_{i=1}^{q} \beta(\pi^*\mathscr{L}, \sum_{j=i}^{q} \tilde{D}_{j,v})\lambda_{\tilde{D}_{i,v}}(p)
\end{aligned}$$

Where $\mathbf{Y}_{i,v}$ is the b-Cartier divisor corresponding to $Y_{i,v}$, $\tilde{D}_{i,v}, i = 1, \ldots, q$ are the divisors come from Definition 4.5 by blowup $Y_{1,v} \supset Y_{2,v} \supset \cdots \supset Y_{q,v}$. The last two equality come from [Voj23, Corollary 6.9.]. Since

$$H^0(X_q, \pi^*\mathscr{L}^N(-m\sum_{j=i}^{q}\tilde{D}_{j,v})) \subseteq H^0(X_q, \pi^*\mathscr{L}^N(-m\tilde{D}_{i,v}))$$

, by the definition of β constant, we have $\beta(\pi^*\mathscr{L}, \sum_{j=i}^{q}\tilde{D}_{j,v}) \leq \beta(\pi^*\mathscr{L}, \tilde{D}_{i,v})$. Hence we get the inequality for some constant C

$$\sum_{v\in S}\sum_{i=1}^{q}\beta(\pi^*\mathscr{L}, \sum_{j=i}^{q}\tilde{D}_{j,v})\lambda_{\tilde{D}_{i,v}}(p) \leq \sum_{v\in S}\sum_{i=1}^{q}\beta(\pi^*\mathscr{L}, \tilde{D}_{i,v})\lambda_{\tilde{D}_{i,v}}(P) + C \quad ,\forall P \in X_q$$

since $\lambda_{\tilde{D}_{i,v}}$ has a lower bound. By [Voj23, Theorem 1.9.],

$$\sum_{v\in S}\sum_{i=1}^{q}\beta(\pi^*\mathscr{L}, \tilde{D}_{i,v})\lambda_{\tilde{D}_{i,v}}(P) \leq (1+\epsilon)h_{\mathscr{L}}(P) + C$$

for some constant C and all P outside a proper Zariski-closed subset of X_q. This proves the theorem of birational version. □

Remark 4.7. *The reason that we should do the main theorem is the following lemma:*

Lemma 4.8 ([HLX24],lemma 3.3)**.** *Let Y be a closed subscheme of a projective variety X. Let D, E be closed subschemes on X weakly intersect properly and suppose that $mY \subset D$ and $nY \subset E$ (as closed subscheme). Let A be a big Cartier divisor on X. Then*

$$\beta(A, Y) \geq m\beta(A, D) + n\beta(A, E)$$

In particular

$$\beta(A, D \cap E) \geq \beta(A, D) + \beta(A, E)$$

When we take $Y_i = \bigcap_{j=1}^{i} D_i$, then we have

$$\beta(A, Y_i) - \beta(A, Y_{i-1}) \geq \beta(A, D_i)$$

Using this lemma, we know that in some sence, this is a generalize of the Vojta's case.

5 Greatest common divisors

The main theorem is poweful, we can use it in many aspects, here we use it to give one kind of GCD theorem.

Before that, we can give an example. This can be calculated trivially by the asymptotic Riemann-Roch theorem.

Example 5.1. *Let D be effective divisors on a projective variety X with dimension n, then*

$$\beta(D, D) = \frac{1}{n+1}$$

Theorem 5.2. *Let $D_1, \ldots, D_{n+1}$ be effective divisors intersecting properly on a projective variety X with dimension n, all defined over a number field K. Suppose that there exist positive integers $a_1, \ldots, a_{n+1}$ such that $a_1D_1, \ldots, a_{n+1}D_{n+1}$ are all numerically equivalent to an ample divisor D. Let $B = \{\bigcap_{i\in I} D_i \mid I \subseteq \{1, \ldots, n\}\}$. Suppose that for $Y_l, Y_k \in B$ and there is $Y_k \cap a_sD_s = Y_l$, and for all $Q \in X(K)$,*

$$\beta(A, Y_l) - \beta(A, Y_k) - \frac{1}{n+1} > 0, \tag{5.1}$$

Let S be a finite set of places of K containing all the archimedean ones and let $\epsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subset X$ such that for any set $R \subset X(K)$ of $(\sum_{i=1}^{n} D_i, S)$-integral points and all but finitely many points $P \in R \setminus Z$, we have

$$h_{D_i \cap D_j}(P) \leq \epsilon h_D(P) \quad \forall i, j$$

Proof. Let $P \in R$. For $v \in S$, let $i_1, \ldots, i_{n+1} \in \{1, \ldots, n+1\}$ be such that

$$\lambda_{a_{i_1}D_{i_1},v}(P) \geq \cdots \geq \lambda_{a_{i_{n+1}}D_{i_{n+1}},v}(P).$$

Then, let $Y_{i_j,v} = \bigcap_{k=1}^{j} a_{i_k,v}D_{i_k,v}$, we have

$$\sum_{v\in S}\sum_{j=1}^{n+1} \lambda_{a_{i_j,v}D_{i_j,v},v}(P) = \sum_{v\in S}\sum_{j=1}^{n+1} \lambda_{Y_{i_j,v},v}(P) + O(1)$$

$$\geq (n+1-\epsilon)h_D(P) + O(1).$$

On the other hand, we have our main theorem

$$\sum_{v\in S}\sum_{j=1}^{n+1}(\beta(A, Y_{i_j,v}) - \beta(A, Y_{i_{j-1},v}))\lambda_{Y_{i_j,v},v}(P) + O(1)$$

$$\leq (1+\epsilon)h_D(P) + O(1).$$

Define

$$\gamma_{i_j,v} := (n+1) \cdot (\beta(A, Y_{i_j,v}) - \beta(A, Y_{i_{j-1},v}) - \frac{1}{n+1}) > 0$$

Combining the above two equations, we have

$$(n+1-\epsilon)h_D(P) + \sum_{v\in S}\sum_{j=1}^{n+1} \gamma_{i_j,v}\lambda_{Y_{i_j,v},v}(P)$$

$$< (n+1+\epsilon)h_D(P) + O(1).$$

Let,

$$\gamma = \min_{v\in S, j} \gamma_{i_j,v}$$

Then,

$$\sum_{v\in S}\lambda_{Y_{i_j,v},v}(P) < \frac{2\epsilon}{\gamma}h_D(P)$$

We conclude that for any $\epsilon > 0$, we have

$$\sum_{v\in S}\lambda_{Y_{i_j,v},v}(P) \geq \min_i \lambda_{D_i,v}(P) \leq \epsilon h_D(P) + O(1).$$

Since R is a set of $(\sum_{i=1}^{n} D_i, S)$-integral points, this is equivalent to

$$h_{D_i\cap D_j}(P) \leq \epsilon h_D(P) + O(1)$$

for all $P \in R \setminus Z$ and for all i, j.

□

Remark 5.3. *The inequality (5.1) slightly strengthens Lemma 4.8, and this is necessary. The reason can be referred to in this paper.[HLX24, Example 5.5]*

6 Degeneracy of integral points

Here we give another use of the main theorem, we will use it to say that under some condition, the integral points on a variety should be degneracy.

Theorem 6.1 (Integral points). *Let X be a projective variety over a number field K with dimension n. Let $D_1, \ldots, D_{n+1}$ be effective Cartier divisors on X satisfied that any n of them are intersect properly and $\bigcap_{i=1}^{n+1} D_i \neq \emptyset$. Suppose that there exist positive integers $a_1, \ldots, a_n$ such that $a_1D_1, \ldots, a_nD_n$ are all numerically equivalent to an ample divisor D. Denote $\bigcap D_i$ by Y Let*

$$b = \frac{1}{\min\limits_{\substack{R\subsetneq T\subsetneq\{1,\ldots,n+1\}\\ \#T=\#R+1=n\\ Q\in Y(\overline{K})}} \{\beta(D, (\cap_{j\in T}D_j)_Q) - \beta(D, \cap_{j\in R}D_j)\}}$$

Furthermore, suppose that for every point $Q \in Y(\overline{K})$ we have

$$\text{for all } Q \in Y(\overline{K}),\text{ and } T_1, T_2 \subsetneq \{1, \ldots, n+1\} \text{ with } \#T_2 = \#T_1 = n,$$
$$(1-b)(\beta(D,(\cap_{j\in T_1}D_j)_Q) - \beta(D, \cap_{j\in T_1\cap T_2}D_j)) + b(\beta(D, (\cap_{j\in T_2}D_j)_Q) - \beta(D, \cap_{j\in T_1\cap T_2}D_j)) > 2$$

Let S be a finite set of places of K containing all the archimedeam places. Then there exists a proper Zariski-closed subset $Z \subset X$ such that for any set $R \subset X(K)$ of $(D_1 + \cdots + D_{n+1}, S)$-integral points, the set $R \setminus Z$ is finite.

Proof. After replacing K by a finite extension, we can assume that every point in the support of $D_{i_1} \bigcap \cdots \bigcap D_{i_n}$ is K-rational.

We first show that for any $\epsilon > 0$ there exists a proper Zariski-closed subset $Z \subset X$ such that for any set $R \subset X(K)$ of $(D_1 + \cdots + D_{n+1}, S)$-integral points, we have

$$\begin{aligned} h_{a_1 D_1 \cap \cdots \cap a_{n+1} D_{n+1}}(P) &= \sum_{v \in S} \lambda_{a_1 D_1 \cap \cdots \cap a_n D_n, v}(P) + O(1) \\ &= \sum_{v \in S} \min_i (\lambda_{a_i D_i}(P)) + O(1) \\ &\leq \epsilon h_D(P) + O(1) \end{aligned}$$

for all $P \in R \backslash Z$. By the definition and elementary properties of heights, for any $\epsilon > 0$,

$$\sum_{v \in S} \sum_{i=1}^{n+1} \lambda_{a_i D_i, v} \geq (3 - \epsilon) h_D(P) + O(1)$$

for all $P \in R$, where $O(1)$ is independent with P.

Let $P \in R$, For each $v \in S$, let $\{i_{1,v}, \ldots, i_{n+1,v}\} = \{1, \ldots, n+1\}$ be such that

$$\lambda_{a_{i_{1,v}} D_{i_{1,v}}, v}(P) \geq \lambda_{a_{i_{2,v}} D_{i_{2,v}}, v}(P) \geq \cdots \geq \lambda_{a_{i_{n+1,v}} D_{i_{n+1,v}}, v}(P)$$

.

For each $v \in S$, there exists

$$Q_v \in \operatorname{Supp}(\cap_{j=1}^n a_{i_{j,v}} D_{i_{j,v}})$$

(depending on P) such that

$$\lambda_{\cap_{j=1}^n a_{i_{j,v}} D_{i_{j,v}}, v} = \lambda_{(\cap_{j=1}^n a_{i_{j,v}} D_{i_{j,v}})_{Q_v}, v} + O(1)$$

Where the constant $O(1)$ is independent of P. If $Q_v \notin \operatorname{Supp}(\cap_{j=1}^{n+1} D_j)$, then

$$\begin{aligned} \lambda_{\cap_{j=1}^{n+1} a_j D_j, v}(P) &= \min(\lambda_{\cap_{j=1}^n a_{i_{j,v}} D_{i_{j,v}}, v}(P), \lambda_{a_{i_{n+1,v}}, v}(P)) \\ &= \min(\lambda_{(\cap_{j=1}^n a_{i_{j,v}} D_{i_{j,v}})_{Q_v}, v}(P), \lambda_{a_{i_{n+1,v}} D_{i_{n+1,v}}, v}(P)) + O(1) \\ &= \lambda_{(\cap_{j=1}^n a_{i_{j,v}} D_{i_{j,v}})_{Q_v} \cap a_{i_{n+1,v}} D_{n+1,v}} \\ &= O(1) \end{aligned}$$

When $Q_v \in \operatorname{Supp}(\cap_{j=1}^{n+1} D_j)$, then we use

$$\lambda_{a_{i_{n+1,v}} D_{i_{n+1,v}}, v}(P) \leq \lambda_{(\cap_{j \neq n} a_{i_{j,v}} D_{i_{j,v}})_{Q_v}, v}(P) + O(1)$$

Let

$$S^{'} = \{v \in S | Q_v \notin \operatorname{Supp} \cap_{j=1}^{n_1} D_j\}$$

$$S^{''} = S \backslash S'$$

It follows that

$$
\begin{aligned}
(3-\epsilon)h_D(P) &\leq \sum_{v\in S}\sum_{i=1}^{n+1}\lambda_{a_iD_i,v} \\
&\leq \sum_{v\in S'}
\end{aligned}
$$

Let Q be some point in $\operatorname{Supp}(\bigcap_{j=1}^{n+1} D_j)$ (which is nonempty by assumption). Let $\pi : \tilde{X} \to X$ be the blowup at Q, with exceptional divisor E. If R is a set of $(\sum_{j=1}^{n+1} D_j, S)$-integral points in $X(K)$, then $\pi^{-1}(R) \setminus E$ is a set of $(\sum_{j=1}^{n+1} \pi^* D_j, S)$-integral points in $\tilde{X}(K)$. So it suffices to show that there exists a proper Zariski-closed subset $\tilde{Z}$ of $\tilde{X}$ such that for any set $\tilde{R}$ of $(\sum_{j=1}^{n+1} \pi^* D_j, S)$-integral points in $\tilde{X}(K)$, the set $\tilde{R} \setminus \tilde{Z}$ is finite.

Define the effective Cartier divisors

$$D'_i = a_i \pi^* D_i - E, \quad i = 1, \cdots, n+1.$$

Let $\tilde{R}$ be a set of $(\sum_{j=1}^{n+1} \pi^* D_j, S)$-integral points in $\tilde{X}(K)$ (and hence a set of $(\sum_{j=1}^{n+1} D'_j, S)$-integral points). For $P \in \tilde{R}$ and $\epsilon > 0$, we have

$$
\begin{aligned}
\sum_{v\in S}\left(\sum_{j=1}^{n+1}\lambda_{D'_j,v}(P)\right) &= \sum_{j=1}^{n+1} h_{D'_j}(P) + O(1) \\
&\geq (3-\epsilon)h_{\pi^*D}(P) - (3-\epsilon)h_E(P) + O(1)
\end{aligned}
$$

where the $O(1)$ possibly depends on $\tilde{R}$ (but not P).

We now bound the left-hand side of the above equation. As in previous arguments, it suffices to bound a sum of the form

$$\sum_{v\in S}\left(\lambda_{D'_{i_{1,v}},v}(P) + \lambda_{D'_{i_{1,v}}\cap D'_{i_{2,v}},v}(P) + \cdots + \lambda_{\bigcap_{j=1}^{n+1} D'_{i_{j,v}},v}(P)\right),$$

where $\{i_{1,v}, \cdots, i_{n+1,v}\} = \{1, \cdots, n+1\}$ for $v \in S$. We first note that it follows from equitionand functoriality that given $\epsilon > 0$, there exists a proper Zariski-closed subset $\tilde{Z} \subset \tilde{X}$ such that

$$\sum_{v\in S}\lambda_{\bigcap_{j=1}^{n+1} D'_{i_{j,v}},v}(P) < \epsilon h_{\pi^*D}(P) + O(1)$$

for all $P \in \tilde{R} \setminus \tilde{Z}$. For the same reasons, we may choose $\tilde{Z}$ so that we also have

$$h_E(P) < \epsilon h_{\pi^*D}(P) + O(1)$$

for all $P \in \tilde{R} \setminus \tilde{Z}$. We can write (as closed subschemes)

$$\bigcap_{j=1}^{n} D'_{i_{j,v}} = Y_{0,v} + Y_{1,v},$$

where $\operatorname{Supp}(\pi(Y_{1,v})) = Q$, $\dim Y_{0,v} = 0$, and $Y_{0,v} \cap E = \emptyset$. We have (for an appropriate $\tilde{Z}$)

$$\sum_{v \in S} \lambda_{Y_{1,v},v}(P) < \epsilon h_{\pi^* D}(P) + O(1)$$

for all $P \in \tilde{R} \setminus \tilde{Z}$, and so

$$\lambda_{\bigcap_{j=1}^n D'_{i_{j,v}},v}(P) \leq \lambda_{Y_{0,v},v}(P) + \epsilon h_{\pi^* D}(P) + O(1)$$

for all $P \in \tilde{R} \setminus \tilde{Z}$.

Let $\delta \in \mathbb{Q}, \delta > 0$, be chosen such that [HLX24, Lemma 3.14]

$$\gamma' = \beta(\pi^* D - \delta E, \pi^* D - E) - \frac{1}{n+1} > 0.$$

Note that

$$\beta(\pi^* D - \delta E, Y_{0,v}) \geq \beta(\pi^* D - \delta E, D'_{i_{n+1,v}}) + \beta(\pi^* D - \delta E, \bigcap_{j=1}^n D'_{i_{j,v}}).$$

This does not follow directly from Lemma 4.8 (since $D'_{i_{n+1,v}}$ and $\bigcap_{j=1}^n D'_{i_{j,v}}$ may not intersect properly above Q, along the component $Y_{1,v}$), but it follows from a slight modification to the proof of that lemma as $D'_{i_{n+1,v}}$ and $\bigcap_{j=1}^n D'_{i_{j,v}}$ intersect properly in a neighborhood of the zero-dimensional closed subscheme $Y_{0,v}$.

Using our main theorem, for any $\epsilon > 0$ we find that for $P \in \tilde{R}$ outside a proper Zariski-closed subset of $\tilde{X}$ (and up to $O(1)$),

$$\left(\frac{1}{n+1} + \gamma'\right) \sum_{v \in S} \left(\lambda_{D'_{i_{1,v}},v}(P) + \lambda_{D'_{i_{1,v}} \cap D'_{i_{2,v}},v}(P) + \cdots + \lambda_{\bigcap_{j=1}^{n+1} D'_{i_{j,v}},v}(P)\right)$$

$$\leq \sum_{v \in S} (\beta(\pi^* D - \delta E, D'_{i_{1,v}}) \lambda_{D'_{i_{1,v}},v}(P) + (\beta(\pi^* D - \delta E, D'_{i_{1,v}} \cap D'_{i_{2,v}}) - \beta(\pi^* D - \delta E, D'_{i_{1,v}})) \lambda_{D'_{i_{2,v}},v}(P)$$

$$+ \cdots + (\beta(\pi^* D - \delta E, \bigcap_{j=1}^n D'_{i_{j,v}}) - \beta(\pi^* D - \delta E, \bigcap_{j=1}^{n-1} D'_{i_{j,v}})) \lambda_{\bigcap_{j=1}^n D'_{i_{j,v}},v}(P) +$$

$$(\beta(\pi^* D - \delta E, Y_{0,v}) - \beta(\pi^* D - \delta E, D'_i) \lambda_{Y_{0,v},v}(P)) + \epsilon h_{\pi^* D}(P)$$
$$\leq (1 + \epsilon) h_{\pi^* D - \delta E}(P) + \epsilon h_{\pi^* D}(P) \leq (1 + 2\epsilon) h_{\pi^* D}(P).$$

Therefore, for some positive $\delta' > 0$, for $P \in \tilde{R}$ outside a proper Zariski-closed subset of $\tilde{X}$ we have

$$\sum_{v \in S} \sum_{j=1}^{n+1} \lambda_{D'_j,v}(P) \leq (3 - \delta') h_{\pi^* D}(P).$$

On the other hand, by the proof above (taking ϵ sufficiently small), for $P \in \tilde{R}$ outside a proper Zariski-closed subset of $\tilde{X}$,

$$\sum_{v \in S} \sum_{j=1}^{n+1} \lambda_{D'_j,v}(P) \geq \left(3 - \frac{\delta'}{2}\right) h_{\pi^* D}(P).$$

Finally, since π^*D is big, combining the above inequalities with an application of Northcott's theorem (for big divisors) gives that there exists a proper Zariski-closed subset $\tilde{Z}$ of $\tilde{X}$ such that $\tilde{R} \setminus \tilde{Z}$ is finite.

□

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References

[Aut11] Pascal Autissier, *Sur la non-densité des points entiers*, Duke Math. J. **158** (2011), no. 1, 13–27. MR 2794367

[Cor07] Alessio Corti, *3-fold flips after Shokurov*, Flips for 3-folds and 4-folds, Oxford Lecture Ser. Math. Appl., vol. 35, Oxford Univ. Press, Oxford, 2007, pp. 18–48. MR 2359340

[CZ02] Pietro Corvaja and Umberto Zannier, *A subspace theorem approach to integral points on curves*, C. R. Math. Acad. Sci. Paris **334** (2002), no. 4, 267–271. MR 1891001

[CZ04] P. Corvaja and U. Zannier, *On integral points on surfaces*, Ann. of Math. (2) **160** (2004), no. 2, 705–726. MR 2123936

[EF08] Jan-Hendrik Evertse and Roberto G. Ferretti, *A generalization of the Subspace Theorem with polynomials of higher degree*, Diophantine approximation, Dev. Math., vol. 16, SpringerWienNewYork, Vienna, 2008, pp. 175–198. MR 2487693

[HL21] Gordon Heier and Aaron Levin, *A generalized Schmidt subspace theorem for closed subschemes*, Amer. J. Math. **143** (2021), no. 1, 213–226. MR 4201783

[HL23] Gordon Heier and Aaron Levin, *A schmidt-nochka theorem for closed subschemes in subgeneral position*, 2023.

[HLX24] Keping Huang, Aaron Levin, and Zheng Xiao, *A new diophantine approximation inequality on surfaces and its applications*, 2024.

[Lev09] Aaron Levin, *Generalizations of Siegel's and Picard's theorems*, Ann. of Math. (2) **170** (2009), no. 2, 609–655. MR 2552103

[RV20] Min Ru and Paul Vojta, *A birational Nevanlinna constant and its consequences*, Amer. J. Math. **142** (2020), no. 3, 957–991. MR 4101336

[Sil87] Joseph H. Silverman, *Arithmetic distance functions and height functions in Diophantine geometry*, Math. Ann. **279** (1987), no. 2, 193–216. MR 919501

[Voj87] Paul Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Mathematics, vol. 1239, Springer-Verlag, Berlin, 1987. MR 883451

[Voj97] ———, *On Cartan's theorem and Cartan's conjecture*, Amer. J. Math. **119** (1997), no. 1, 1–17. MR 1428056

[Voj23] ———, *Birational Nevanlinna constants, beta constants, and diophantine approximation to closed subschemes*, J. Théor. Nombres Bordeaux **35** (2023), no. 1, 17–61. MR 4596522